THE K-FACTOR GARMA PROCESS WITH INFINITE VARIANCE INNOVATIONS

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Abstract. In this article, we develop the theory of k-factors Gegenbauer Autoregressive Moving Average (GARMA) process with infinite variance innovations. We establish conditions for existence and invertibility of the model. We also discuss the parameter estimation by using two methods. The first one is the Conditional Sum of Squares (CSS) approach and the second is the Markov Chains Monte Carlo (MCMC) Whittle method. For comparison purpose, Monte Carlo simulations are used to evaluate the finite sample performance of these estimation techniques.

Keywords. Stable distributions, Gegenbauer polynomial, Long memory.

1 Introduction

When dealing with empirical time series arising from diverse fields of applications, we are confronted with the phenomenon of long memory or long-range dependence. A time series with this property has a slow and hyperbolically declining autocorrelation function or, equivalently an infinite spectrum at zero frequency. A popular way to analyze a long memory time series is to use fractionally integrated autoregressive moving average (FARIMA) processes introduced by Granger and Joyeux (1980) and Hosking (1981). They assume in their theory that the innovations are gaussian. However, we realize that this hypothesis is too restrictive, particularly, in some domains such as finance or telecommunication in which one must take into account a high variability of the data which is translated by infinite variance. To achieve this gaol, Kokoszka and Taqqu (1994) have introduced the FARIMA processes with infinite variance innovations helping to take into account the behavior of long memory and infinite variance. Furthermore, most of time series in real life may have a persistent periodic behavior, in addition to long term structure. Unfortunately, the FARIMA model does not allow to take into account a periodic or cyclical behavior. Thus, the methodology for modeling time series with long memory behavior has been extended to long memory time series with seasonal components. Recent contributions related to the seasonal FARIMA model (hereafter denoted ARFISMA model) are Porter-Hudak (1990), Hassler (1994), Arteche and Robinson (2000) and Reisen et al (2004). In recent years, Diongue et al (2008) have developed the theory of ARFISMA model with stable innovations which allow to take the presence of long memory, seasonality and infinite variance. It is worthwhile to note that the seasonal models mentioned above represent a special case of the stable ARFISMA models. However, these models are very limited, insofar as they consider that the seasonal frequencies are fixed and known. In fact, most of time series in real life, exhibit long-memory periodical bihavior at any frequency of the spectrum. See for examples Gray et al (1989), Woodward et al (1998), Giraitis and Leipus (1995) and Chung (1996) for environmental data and Ferrara and Guégan (1999) for real data of traffic in the subway of Paris. Thus, we developed in this paper the theory of k-factor Gegenbauer Autoregressive Moving-Average (GARMA) processes with stable innovations which allow the modeling of long memory data containing seasonal periodicities and infinite variance components.

2 The model

2.1 Stable distributions

In this section, we summarize the relevant facts associated with the stable distributions and refer the reader to Samorodnitsky and Taqqu (1994) for a detailed statistical description. There are several ways of defining the stable distributions. In general, it will be convenient to define them in terms of their characteristic functions. A random variable X is said α-stable, denoted $S_{\alpha,\beta}(\mu,\sigma)$, if its characteristic function is given by

$$\Phi_X(t) = \begin{cases} \exp\left\{i\mu t - \sigma^{\alpha}|t|^{\alpha}\left[1 - i\beta \operatorname{sign}(t)\tan\frac{\pi\alpha}{2}\right]\right\} & \text{if } \alpha \neq 1, \\ \exp\left\{i\mu t - \sigma|t|\left[1 + i\beta\frac{2}{\pi}\operatorname{sign}(t)\ln|t|\right]\right\} & \text{if } \alpha = 1, \end{cases}$$

Notice that the parameter α ($0 < \alpha \leq 2$) is called the index of stability or characteristic exponent, the true value β ($-1 \leq \beta \leq 1$) is a measure of departure from symmetry, while the parameters $\sigma > 0$ and μ ($-\infty < \mu < +\infty$) are the scale parameter and the location parameter respectively.

In the following of this paper, we will consider the symmetric α-stable distribution that we denote SαS.

2.2 The k-factor GARMA(p, d, ν, q)-SαS process

We introduce here the model we will work with. Assume that $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of independently and identically distributed (i.i.d.) SαS ($0 < \alpha \leq 2$) random variables with mean zero and scale parameter equal to 1. Let $\Phi(B) = 1 - \sum_{j=1}^{p}\phi_j B^j$ and $\Theta(B) = 1 + \sum_{j=1}^{q}\theta_j B^j$ denote the ARMA operators and have no common roots. Assume that all the roots of the polynomials $\Phi(B)$ and $\Theta(B)$ lie outside the unit circle.

We define a centered k-factor GARMA process $(X_t)_{t\in\mathbb{Z}}$ with SαS innovations by

$$\Phi(B)\prod_{j=1}^{k}(I - 2\nu_j B + B^2)^{d_j} X_t = \Theta(B) Z_t \tag{1}$$

where k is a non negative integer, $|\nu_i| \leq 1$ for $i = 1,\ldots,k$ and $d_j \neq 0$ ($j = 1,\ldots,k$) are (fractional) differencing degrees. The frequencies $\lambda_j = \arccos(\nu_j)$ for all $j = 1,\ldots,k$ are called the Gegenbauer frequencies (or G-frequencies). We recall that the Gegenbauer polynomials, often used in applied mathematics because of their orthogonality and recursion properties, are defined by:

$$\left(1 - 2\nu z + z^2\right)^{-d} = \sum_{j\geq 0} C_j(d,\nu) z^j, \tag{2}$$

where $|z| \leq 1$ and $|\nu| \leq 1$. A more easy way to compute the Gegenbauer polynomials $\left(C_j(d,\nu)\right)_{j\geq 0}$ is based on the following recursion formula:

$$\begin{cases} C_0(d,\nu) = 1 \\ C_1(d,\nu) = 2d\nu \\ C_j(d,\nu) = 2\nu\left(\frac{d-1}{j} + 1\right) C_{j-1}(d,\nu) - \left(2\frac{d-1}{j} + 1\right) C_{j-2}(d,\nu),\ \forall j > 1. \end{cases} \tag{3}$$

In the following theorem, we establish the invertibility and stationarity conditions of model (1).

Theorem 1: Let $d_i \neq 0$ for $i = 1,\ldots,k$. The k-factor GARMA(p, d, ν, q)-SαS process $(X_t)_{t\in\mathbb{Z}}$ defined by equation (1) have the following properties:

i) The process $(X_t)_{t\in\mathbb{Z}}$ is stationary if

$$d_i < \begin{cases} 1-\frac{1}{\alpha} & \text{if } |\nu_i|<1 \\ \frac{1}{2}\left(1-\frac{1}{\alpha}\right) & \text{if } |\nu_i|=1 \end{cases} \tag{4}$$

ii) The process $(X_t)_{t\in\mathbb{Z}}$ is invertible if

$$d_i > \begin{cases} -1+\frac{1}{\alpha} & \text{if } |\nu_i|<1 \\ -\frac{1}{2}\left(1-\frac{1}{\alpha}\right) & \text{if } |\nu_i|=1, \end{cases} \tag{5}$$

iii) The power transfer function of $(X_t)_{t\in\mathbb{Z}}$ is given by:

$$f_X(\omega)=\frac{|\Theta\left(e^{-i\omega}\right)|^2}{|\Phi\left(e^{-i\omega}\right)|^2}\prod_{j=1}^{k}\left|4\sin\left(\frac{\omega+\lambda_j}{2}\right)\sin\left(\frac{\omega-\lambda_j}{2}\right)\right|^{-2d_j},\ -\pi\le\omega\le\pi. \tag{6}$$

Under conditions (4) and (5), the AR(∞) and MA(∞) representations are respectively:

$$Z_t=\sum_{j=0}^{\infty}\alpha_j(d,\nu,\phi,\theta)X_{t-j}\qquad X_t=\sum_{j=0}^{\infty}\beta_j(d,\nu,\phi,\theta)Z_{t-j},$$

where $d=(d_1,\ldots,d_k)$, $\nu=(\nu_1,\ldots,\nu_k)$, $\phi=(\phi_1,\ldots,\phi_p)$ and $\theta=(\theta_1,\ldots,\theta_q)$. The coefficients $\beta_j(d,\nu,\phi,\theta)$ and $\alpha_j(d,\nu,\phi,\theta)$ can be determined by:

$$\Phi(z)\sum_{j=0}^{\infty}\beta_j(d,\nu,\phi,\theta)z^j=\Theta(z)\sum_{j=0}^{\infty}\psi_j(d,\ \nu)z^j \tag{7}$$

$$\Theta(z)\sum_{j=0}^{\infty}\alpha_j(d,\nu,\phi,\theta)z^j=\Phi(z)\sum_{j=0}^{\infty}\pi_j(d,\ \nu)z^j \tag{8}$$

where the weights $\psi_j(d,\nu)$ and $\pi_j(d,\nu)$ are given by:

$$\psi_j(d,\nu)=\sum_{\substack{0\le l_1,\cdots,l_k\le j,\\ l_1+\cdots+l_k=j}}C_{l_1}(d_1,\nu_1)\cdots C_{l_k}(d_k,\nu_k)\qquad\text{and}\qquad \pi_j(d,\nu)=\psi_j(-d,\nu), \tag{9}$$

and the weights $(C_{l_i}(d,\nu))_{l_i\in\mathbb{Z}}$ are the Gegenbauer polynomials previously defined.

3 Estimation methods

In this Section, we are interested in the estimation of the parameters of a k-factor GARMA($p,\ d,\ \nu,\ q$)-SαS process by using the Markov Chains Monte Carlo (MCMC) Whittle method (e.g. Ndongo et al (2010)) and the Conditional Sum of Squares (CSS) procedure. Let $X_1,\ldots,X_T$ be an observed finite sequence generated by a symmetric α-stable causal stationary invertible k-factor GARMA process $(X)_{t\in\mathbb{Z}}$ defined by equation (1). Assume that all the Gegenbauer frequencies are known and denote by $\psi=(\phi,\ \theta,\ d)$ the vector of parameters of interest, where $\phi=(\phi_1,\ldots,\phi_p)$ and $\theta=(\theta_1,\ldots,\theta_q)$ are the coefficients of the autoregressive polynomial $\Phi(z)$ and the moving average polynomial $\Theta(z)$ respectively, d is the k-vector $(d_1,\ldots,d_k)$. We assume that $\psi_0=(\phi_0,\ \theta_0,\ d_0)$ is the true value of ψ and that ψ_0 is in the interior of the compact set $\Psi\subseteq\mathbb{R}^{p+q+k}$.

3.1 The MCMC Whittle method

In the particular case of the ARFISMA-SαS model, Ndongo et al (2010) have developed the MCMC Wittle procedure to estimate the parameters. It is based on approximation of the Whittle likelihood function using the MCMC method. Here, we will consider this approach to estimate the parameters of k-factor GARMA$(p,\ d,\ \nu,\ q)$-SαS process.
The MCMC Whittle's estimator $\widehat{\psi}_W$ of $\psi_0 = (\phi_0,\ \theta_0,\ d_0)$ is obtained by minimizing the following likelihood function:

$$L_W\left(X,\ \psi\right) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{f_X(\omega_j,\psi)},$$

where N is taken large enough from the strong law of large number and $f_X(\omega,\psi)$ is the power transfer function of the process $(X_t)_{t\in\mathbb{Z}}$ generating the data. The sequence $\omega_1,\ldots,\omega_N$ is generated using a Metropolis-Hastings algorithm.

3.2 The CSS method

Chung and Baillie (1993) proposed a method based on the minimization of conditional sum of squared residuals. As remarked by Chung (1996a), it is important to note that the normality assumption is not essential in the CSS estimation. Here, we generalize this estimator to k-factor GARMA -SαS models. Thus, the conditional sum of squares estimator $\widehat{\psi}_{CSS}$ of ψ_0 is the value of ψ which minimizes

$$S(\psi) = \sum_{t=1}^{T}\left[Z_t\left(\psi\right)\right]^2.$$

However, given the observation $X_1,\ldots,X_T$ the innovation $(Z_t)_{t=1,\ldots,T}$ cannot be directly computed, since an infinite sample would be needed. Nevertheless, they may be estimated by:

$$Z_t\left(\psi\right) = \sum_{j=0}^{T-1}\alpha_j(\psi)X_{t-j}, \qquad t = 1,\ldots,T,$$

where the coefficients $\left(\alpha_j(\psi)\right)_{j\geq 0}$ are defined by equation (8).

4 Monte Carlo simulations

In this section, we focus our attention to Gegenbauer processes with SαS innovations, which are GARMA processes without the autoregressive and moving-average parts. Thus, it is a special case $k=1$ and $q=p=0$ of model (1). In addition, if $\alpha=2$ then we obtain the Gaussian Gegenbauer processes introduced by Gray et al (1989). We propose in this section to estimate the long memory parameter of Gegenbauer processes with SαS innovations, using the CSS method and compared it with the MCMC Whittle method. Thus, in this experiment we consider a SαS Gegenbauer process with $d=0.3$, $\nu=0.8$ and $\alpha=1.2$ and we consider three sample sizes $T=250$, $T=500$ and $T=1000$. The simulation results give the average values, the root mean square error (RMSE) and the mean absolute error (MAE) of the estimation procedures based on 1500 replications. The results are summarized in Table 1. It can be seen from this table that all methods have good performance (small bias and RMSE), even for small sample size. It seems that better estimates are obtained from the CSS method than the MCMC Whittle procedure. The simulation results also show the impact of the sample size T on these estimation methods (when T increases, the results improves).

Table 1: Monte Carlo study to compare CSS method with MCMC-Whittle method for SαS Gegenbauer process with $d = 0.3$, $\nu = 0.8$ and $\alpha = 1.5$.

	$T = 250$		$T = 500$		$T = 1000$	
Statistics	$\hat{d}_{CSS}$	$\hat{d}_W$	$\hat{d}_{CSS}$	$\hat{d}_W$	$\hat{d}_{CSS}$	$\hat{d}_W$
Mean	0.3103	0.3089	0.3085	0.3080	0.3067	0.3068
RMSE	0.0264	0.0278	0.0216	0.0235	0.0170	0.0190
MAE	0.0223	0.0236	0.0176	0.0192	0.0131	0.0154

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