The W_N – statistic: a robust nonparametric test statistic

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Abstract

The failure of the F – test statistic and the Kruskal – Wallis (KW) H – test statistic to immediately identify the treatments that are responsible for the rejection of the null hypothesis of no difference can be seen as a shortcoming. However, very many post – hoc (i.e., multiple comparison) methods have been introduced to tackle this deficiency. Some of these also have their limitations especially when the power of the tests is considered since the error rates may differ. Statisticians have been living with these shortcomings and limitations. The work in this paper is an attempt at dealing with the shortcomings of the F – test, the H – test as well as the Friedman's test for the one – way and two – way (or the single factor and two factors) analysis of variance.

Keywords: F – test statistic, Friedman's test statistic, Kruskal – Wallis H – test statistic, W_N – statistic.

1. Introduction

Consider a client (or investigator) faced with the problem of the one – or two – way analysis of variance of a sample obtained from an experiment. Suppose the client approached a statistician to do the analysis. If the null hypothesis of no difference is rejected in either the one – or two – way analysis of variance experiment, the statistician may stop at that point. Nonetheless, the client's problem is not really solved until the statistician identifies the treatments that are significantly different from others. It is against this background that we develop the W_N – statistic (or simply the W_N). The W_N –statistic is robust in that it does not only give the same decision as the F – test, the KW H – test and the Friedman's test under normal circumstances, but also goes beyond the existing shortcomings of the F – test, the KW H – test and the Friedman's test statistics.

2. Development of the W_N – Statistic

A nonparametric multiple-testing procedure that takes into consideration all observations of the combined $p \geq 2$ treatments in a one-way ANOVA problem is developed and called the W_N – statistic. The procedure is based on functions of ranks. In a sense, the method stands out as a nonparametric analogue of the Duncan multiple range test as well as a generalization of the nonparametric Tukey's test. The distribution (both the exact and the asymptotic) of the W_N test statistic is given in section 2.1. The asymptotic relative efficiency for the W_N statistic is theoretically established to be $\frac{3}{\pi}$, which is about 95.5% when compared with the Tukey's test (see Aiyelo and Ogbonmwan, 1994).

Both the empirical and theoretical substantiation of the conjecture that the W_N-statistic is conservative for unbalanced sample size cases are considered and proved. Finally, the exact (permutation) distribution of the W_N-statistic is established. This leads to the development of a statistical table for the W_N-test statistic.

2.1. Methodology

For a class of rank order statistics, let

$$X_i = \left(x_{i1}, x_{i2}, \ldots, x_{in_k}\right), \quad k = 1, 2, \cdots, p,$$

and

$$E_N = \left(E_{N,1}, E_{N,2}, \ldots, E_{N,p}\right) \tag{2.1}$$

where $$E_{N,\alpha} = J_N\left(\frac{\alpha}{N+1}\right) = \frac{\alpha}{N+1} \qquad 1 \leq \alpha \leq N \tag{2.2}$$

$N = n_1 + n_2 + \ldots + n_p$, and $J_N(.)$ is defined as in Chernoff and Savage (1958).

The mean is given as

$$\overline{E}_N = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha},$$

while the variance, A_N^2, is expressed as

$$A_N^2 = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}^2 - \overline{E}_N^2 .$$

Consider the rank order statistic, $h_N(x_i)$, of the form:

$$h_N(x_i) = \frac{1}{n_i}\sum_{\alpha=1}^{N} E_{N,\alpha} Z_{N,\alpha}^{(i)} \quad , \quad i = 1,2,..., p \tag{2.3}$$

where $Z_{N,\alpha}^{(i)} = 1$, if the α th smallest observation (in the combined ranking of all the N observations) is from the ith sample and $Z_{N,\alpha}^{(i)} = 0$ otherwise. Then, after some algebra, we derive the test statistic:

$$W_N = \max_{1 \le i,\, j \le p} \left[n^{1/2} A_N^{-1} \left| h_N(x_i) - h_N(x_j) \right| \right] . \tag{2.4}$$

Further simplification of equation (2.4) yields

$$W_N = n^{1/2} A_N^{-1} \left[\max\ h_N(x_i) - \min\ h_N(x_i) \right] ,\ \ i = 1,\, 2,...\, p\ . \tag{2.5}$$

2.2 Distribution of the W_N-statistic

2.2.1 Exact distribution of W_N

For small samples, no suitable algebraic expression is feasible for the distribution of W_N. However, a permutation method of evaluating the exact distribution of W_N is realizable. This involves a lot of computational complexities. Suppose a realized data set of the form

$$X_N = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix},$$

is composed of N independent identically distributed random variables. Then, there exist $\dfrac{N!}{\prod_{i=1}^{p} (n_i)!}$ possible permutations of the N variables partitioned into p-subsets of size $n_i \quad (i = 1,\, 2,...\, p)$. Consider the set of all these permutations. For each, compute the value of W_N. Then the probability of the different values of W_N which eventually yields the exact permutation of W_N can be obtained.

2.2.2 Asymptotic distribution of W_N

For large samples, consider the following theorem:

Theorem 1 (Ogbonmwan, 1983)**:** Assuming that equations (2.1 – 2.6) are satisfied then under the null hypothesis that:

$$H_0 : F_1(x) = F_2(x) = ... = F_p(x) \quad \forall\ x$$

$$\lim_{N \to \infty} prob\ \{W_N \leq t\} = \chi_p^{(t)}$$

where $\chi_p^{(t)}$ is the cdf of the sample range in a sample of size p drawn from a standardized normal distribution.

From the proof of the theorem, we conclude that for a pre-assigned level of significance α_0,

$$prob\ \{W_N < W_{N,p}(\alpha_0)\} = 1 - \alpha_0$$

where $W_{N,p}(\alpha_0)$ is the upper $100\,\alpha_0\%$ point of the exact null distribution of W_N.

2.3 Testing procedure

The multi-sample testing procedure is formulated as follows:

1. For a realized data set, compute W_N.
2. Compute the value of $W_{N,p}(\alpha_0)$ corresponding to the pre-assigned level of significance, α_0. But due to Theorem 1, $W_{N,p}(\alpha_0) \to R_p(\alpha_0)$. So we read the value of $R_p(\alpha_0)$ from the tables of upper $100\,\alpha_0\%$ point of the exact null distribution of $\chi_p^{(t)}$. For small sample values, read the value in Ogbonmwan and Odiase (2004).
3. Regard the p treatment to be significantly different if $W_N \geq W_{N,p}(\alpha_0)$.
4. If the null hypothesis of "no difference" between the p treatments is rejected, then test using steps (1), (2) and (3) above, the sets of $(p-1), (p-2), ..., 2$ treatments all taken from the p – treatments. If any set of $k \leq p$ treatment is found not to be significantly different, then any subset of the k treatments is also not significantly different.

2.4 Computational analysis and consideration for ties

This section contains the computational aspects in the use of the W_N – statistics with and without ties in the samples as well as the case of unbalanced sample size.

2.4.1 Without ties

When ties do not exist in the distribution, $E_{N,\alpha}$ simply becomes

$$E_{N,\alpha} = \frac{\alpha}{N+1}, \quad \text{with} \quad 1 \le \alpha \le N$$

Hence,

$$\overline{E}_N = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{\alpha}{N+1} = \frac{1}{2}$$

Also, $\sum_{\alpha=1}^{N} E_{N,\alpha}^2 = \frac{1}{(N+1)^2}\sum_{\alpha=1}^{N}\alpha^2 = \frac{N(2N+1)}{6(N+1)})$

Substitution yields

$$A_N^2 = \frac{N-1}{12(N+1)}$$

$$\Rightarrow A_N^{-1} = \left(\frac{12(N+1)}{N-1}\right)^{1/2}$$

$\therefore$ W_N becomes

$$W_N^* = \max_{1\le i,\, j\le p}\left[\left\{\frac{12n(N+1)}{N-1}\right\}^{1/2}\left|h_N(x_i) - h_N(x_j)\right|\right]$$

2.4.2 With ties

If t is the number of tied observations in a group then A_N^2 will be reduced by

$$\frac{1}{N}\cdot\frac{1}{(N+1)^2}\sum\frac{T}{12} \quad \left(= \frac{\sum T}{12N(N+1)^2}\right)$$

where $\sum T$ is the sum of the values of $T = t(t-1)(t+1)$ for all groups of ties.

By substitution, A_N simplifies to:

$$A_N^2 = \frac{N(N^2-1) - \sum T}{12N(N+1)^2}.$$

Hence W_N becomes:

$$W_N^{**} = \max_{1 \le i,\, j \le p} \left[\left\{ \frac{12\, nN\,(N+1)^2}{N\,(N^2-1) - \sum T} \right\}^{1/2} \left| h_N(x_i) - h_N(x_j) \right| \right]$$

2.5. The case of unbalanced sample size

For unbalanced sample cases, W_N could be modified by replacing it with:

$$W_N^{0} = \max_{1 \le i,\, j \le p} \left[A_N^{-1} \left\{ \frac{n_i + n_j}{2} \right\}^{1/2} \left| h_N(x_i) - h_N(x_j) \right| \right]$$

or

$$W_N^{00} = \max_{1 \le i,\, j \le p} \left[A_N^{-1} \left\{ \frac{2}{n_i + n_j} \right\}^{1/2} \left| h_N(x_i) - h_N(x_j) \right| \right]$$

Both W_N^{0} and W_N^{00} have been shown to be empirically and theoretically conservative (Ogbonmwan and Ozobokeme, 1997; Ogbonmwan and Aitusi, 1999).

In order to make the W_N statistic to be a complete test statistics, a statistical table is developed for it. Ogbonmwan and Odiase (2004) gave the computer Algorithms for the development of the statistical table for the W_N statistics.

3. The two – way ANOVA

This section focuses on the W_N – test for two factors with no interactions.

3.1 The W_N – test for two factors ANOVA (with no interactions)

Consider the case in which factor A (rows) consists of m levels and factor B (columns) consists of n levels so that we have exactly one observation per cell. In this case, there are altogether N = mn experimental units. Thus, x_{ij} is considered as the response for the i^{th} treatment in the j^{th} block for i = 1,2,3, … ,m and j = 1,2,3, … , n.

Let $X_{ij} = \{x_{ij}\}$, i = 1, 2, 3, … , m, j = 1, 2, 3, … , n.

$$E_N = \left(E_{N,1}, E_{N,2}, \ldots, E_{N,N}\right), \quad N = mn \tag{3.1}$$

where $E_{N,\alpha} = J_N\left(\frac{\alpha}{N+1}\right) = \frac{\alpha}{N+1} \qquad 1 \le \alpha \le N$

and $J_N(.)$ is defined as in Chernoff and Savage (1958).

Recall that

$\overline{E}_N = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}$, and $A_N^2 = \frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}^2 - \overline{E}_N^2$.

Let $Z_{N,\alpha}^{(i)} = 1$ if the αth smallest observation in the combined ranking of all the N observations is from the ith row and let $Z_{N,\alpha}^{(i)} = 0$ otherwise, for α = 1, 2, 3, … , N. Similarly, let $Z_{N,\alpha}^{(j)} = 1$ if the αth smallest observation in the combined ranking of all the N observations is from the jth column and let $Z_{N,\alpha}^{(j)} = 0$ otherwise, for α = 1, 2, 3, … , N.

Consider the following rank statistics:

$$h_N(x_{i\bullet}) = \frac{1}{m}\sum_{\alpha=1}^{N} E_{N,\alpha} Z_{N,\alpha}^{(i)} , \quad i = 1,2,3, \ldots , n \tag{3.2}$$

$$h_N(x_{\bullet j}) = \frac{1}{n}\sum_{\alpha=1}^{N} E_{N,\alpha} Z_{N,\alpha}^{(j)} , \quad j = 1,2,3, \ldots , m \tag{3.3}$$

From the works of Chernoff and Savage (1958) and Puri (1964), we have that both $h_N(x_{i\bullet})$ and $h_N(x_{\bullet j})$ are asymptotically normally distributed.

More so, for the treatments (Factor A)

$$W_{Na\bullet} = \max_{1\le i,k\le m}\left[n^{1/2} A_N^{-1}\left|h_N(x_{i\bullet}) - h_N(x_{k\bullet})\right|\right] \tag{3.4}$$

i.e.,

$$W_{Na\bullet} = n^{1/2} A_N^{-1}\left[\max\ h_N(x_{i\bullet}) - \min\ h_N(x_{i\bullet})\right] \tag{3.5}$$

For the blocks (Factor B)

$$W_{N\bullet b} = \max_{1\le j,h\le m}\left[m^{\frac{1}{2}} A_N^{-1} \left| h_N\left(x_{\bullet j}\right) - h_N\left(x_{\bullet h}\right)\right|\right] \tag{3.6}$$

i.e.,

$$W_{N\bullet b} = m^{\frac{1}{2}} A_N^{-1}\left[\max\; h_N\left(x_{\bullet j}\right) - \min\; h_N\left(x_{\bullet j}\right)\right]. \tag{3.7}$$

Theorem 2: Assuming that equations (3.1 – 3.7) are satisfied then under the null hypothesis that:

(i) $H_0 : F_{1a}(x) = F_{2a}(x) = \ldots = F_{pa}(x) \quad \forall\; x$

$$\lim_{N\to\infty} prob\left\{W_{Na\bullet} \le t\right\} = \chi_p^{(t)}$$

(ii) $H_0 : F_{1b}(x) = F_{2b}(x) = \ldots = F_{pb}(x) \quad \forall\; x$

$$\lim_{N\to\infty} prob\left\{W_{N\bullet b} \le t\right\} = \chi_p^{(t)}$$

where $\chi_p^{(t)}$ is the cdf of the sample range in a sample of size p drawn from a standardized normal distribution.

Proof:

The proof follows the adaptation of the proof of Theorem 1.

3.2. Testing procedure

The testing procedure is analogous to that of the W_N - test for the one – way ANOVA. Thus, we simply replace W_N by $W_{Na\bullet}$ or $W_{N\bullet b}$ in sub-section 2.3.

4. Conclusion

In brief, this paper knits together several published works of the author and some of his students. It starts with the pioneering work on the theory of W_N – statistic by the author (Ogbonmwan, 1983) and then navigates through the consideration of efficiency, the numerical substantiation, the conjecture of being conservative and ending with the computer algorithm for the development of a Statistical Table for it. Thus, this paper is a formal presentation of the W_N – statistic as a multisample testing procedure. In sum, the W_N – statistic is designed to meet the needs of clients who desire to know the treatments that differ from the others when the null hypothesis of ‘‘no difference’’ is rejected. It is highly efficient and handles both balanced and unbalanced cases.

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