Parametric and Non-parametric estimations for spatial models

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Plan

Spatial data

Spatial data are collected from different spatial locations on the earth, as in a variety of fields, including soil science, geology, oceanography, econometrics, epidemiology, environmental science,...

- Precipitations at some stations
- Distribution of fish species
- Diagnostic of Malaria

Spatial statistics : *observations are spatially dependent*.

Spatial data

Let S a spatial set, $S \subset \mathbb{R}^N$, $N \geq 2$.

Spatial data can be modeled as realization of random fields $Z = \{Z_s,\ s \in S\}$.

Type of spatial data

Three types :

- ☛ Lattices data
- ☛ Point spatial processes
- ☛ Geostatistical data

The distinction between these types is not always simple

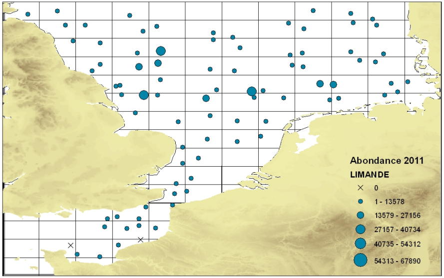
Geostatistic data

Origine : D.G. Krige + Matheron :

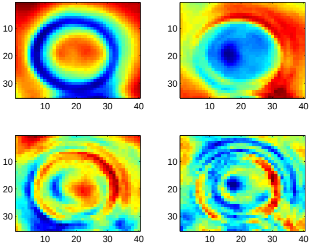
- Data on a continuous set $S \subset \mathbb{R}^d$.
- Measures on a set of sites : $Z = (Z_s,\ s \in S)$ is observed for n fixed sites $\{s_1,\ s_2,\ ...,\ s_n\} \subset S$.

Example : fish species

- Annual collection of data (IBTS : International Bottom Trawl Survey) :
- Observations of fish abundance for some sites of the north sea (France, Belgium,...)
- Prediction on the non-observed sites : fish species maps



Images of malaria-infected erythrocytes



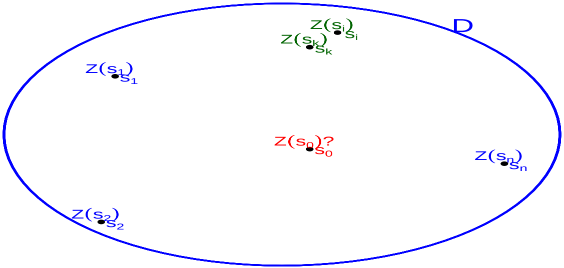
Geostatistics

The literature on spatial models is relatively abundant, see for example Guyon (1995), Anselin and Florax (1995), Cressie (1991) or Ripley (1981), Cressie and Wikle (2011),...

- Spatial variability
- Interpolation : prediction

Let $Z = \{Z_s,\ s \in S\}$ be a real gaussian spatial process that we suppose second order stationary. $Z = (Z_s,\ s \in S)$ is observed on n sites $s_1, ..., s_n$.

Problem

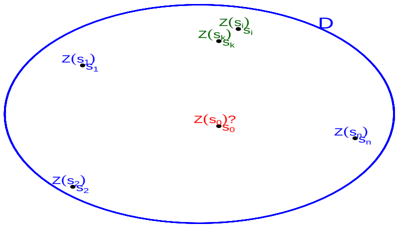


Prediction of Z_{s_0} (or $g(Z_s)$) using $Z_{s_1}, Z_{s_2}, \ldots, Z_{s_n}$, the estimation of the covariance or variogram function

Kriging

- **Model** : stationary spatial process $\{Z_s, s \in D \subset \mathbb{R}^d\}$
- **Data** : sample observed on n sites $s_1, s_2, \ldots, s_n$
- **Variogram :** $\gamma(h)$ (or covariance function $C(h)$) supposed known : $\gamma(h) = \frac{1}{2}\mathrm{Var}(Z_{s+h} - Z_s)$

Prediction of Z_{s_0}



Kriging of Z_{s_0} : BLUP (best linear unbiased predictor) of Z_{s_0} using $\mathcal{Z} = (z_{s_1}, \ldots, z_{s_n})$.

Three types of kriging

- Simple : mean and variance-covariances known.
- Ordinary : $Z_s = \mu + \delta_s$; (δ_s) is a centered stationary process, δ unknown.
- Universal :

$$E(Z_s) = \mu(s),\ \mu(s) = \sum_{j=1}^{p+1} \beta_{j-1} f_{j-1}(s)$$

(β_j) unknown

$$Z_s = \mu(s) + \delta(s) \qquad (\delta_s) \text{ centered stationary}$$

Ordinary Kriging

$$\hat{Z}_{s_0} = \sum_{i=1}^{n} \lambda_i Z_{s_i}$$

with

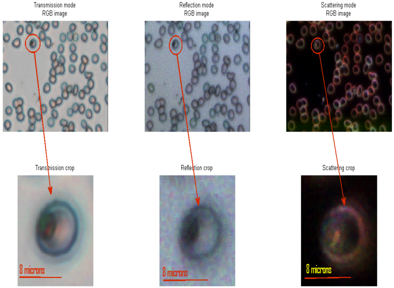
$$E(\hat{Z}_{s_0}) = E(Z_{s_0}) \text{ and } E(\hat{Z}_{s_0} - Z_{s_0})^2 \text{ minimal}$$

The solutions $(\lambda_i)_{i=1,n}$:

$$\begin{pmatrix} 0 & \gamma(s_1 - s_2) & \dots & \gamma(s_1 - s_n) & 1 \\ \gamma(s_1 - s_2) & 0 & \dots & \gamma(s_2 - s_n) & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \gamma(s_1 - s_n) & \gamma(s_1 - s_n) & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \\ m \end{pmatrix} = \begin{pmatrix} \gamma(s_0 - s_1) \\ \gamma(s_0 - s_2) \\ \dots \\ \gamma(s_0 - s_n) \\ 1 \end{pmatrix}$$

Application to Malaria in Africa

- Malaria causes sickness and death in developing countries
- It leads to the highest rate of mortality in the population of children under age 5 years (World Health Organization)
- Contents of malaria-infected erythrocytes analysis, with optical microscopy, depend on the spatial resolution
- No microscope can resolve optical details smaller than half the wavelength of the light used for illumination, Abbe (1873).
- Kriging : a good interpolation method to increase the sampling rate for better perceptibility, and in some cases to filter noise in sampled images (Sanchez, 2005 and Leung, 2001).
- Malaria data obtained with an optique microscope multispectral LED
- 13 images with waves from 375nm to 940nm (transmission, reflection, scattering).


Transmission mode
RGB image
Reflection mode
RGB image
Scattering mode
RGB image
Transmission crop
Reflection crop
Scattering crop
8 microns
8 microns
8 microns

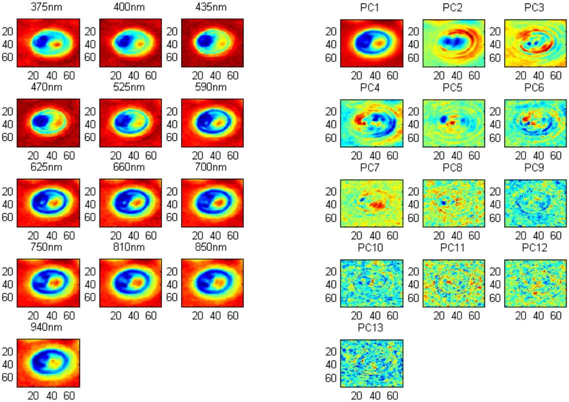


Figure: Original Images and PCA for Transmission

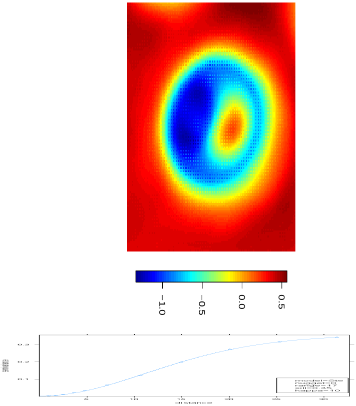


Figure: Kriging of the first PCA Transmission

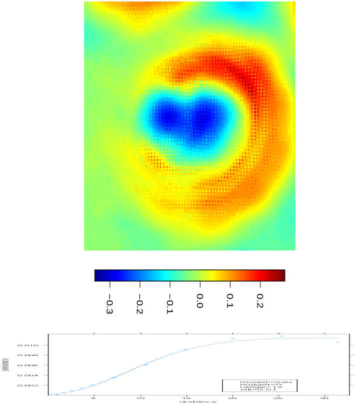


Figure: Kriging of the second PCA Transmission

Spatial prediction : Spatial regression for functional variables

Nonparametric spatial modeling

- ☛ The literature on non-parametric spatial models is less extensive than parametric one : first result Tran (1990).
- ☛ Alternatives to parametric modeling
- ☛ Many potential applications

Spatial prediction : Spatial regression for functional variables

Nonparametric spatial kernel density, regression estimations and prediction

- **Kernel density estimation for discrete-space fields** : See Tran (1990), Carbon, Hallin and Tran (1996), Carbon, Tran and Wu (1997), Lu and Chen (2002, 2004), Hallin, Lu and Tran (2004a), Carbon (2006), Carbon et al. (2009) for real data.
  Basse et al. (2008), Dabo-Niang et al. (2012), Dabo-Niang et Yao (2012),...for functional data

- **Kernel regression estimation, variogram and prediction for discrete-space fields** :
  Biau and Cadre (2004), Hallin, Lu and Tran (2004b), Carbon et al. (2007), Wang and Wang (2009), Menezes et al. (2007, 2010), Wang et al. (2012), for real data.
  Dabo-Niang, Rachdi and Yao (2012) for functional data.

Spatial prediction : Spatial regression for functional variables

- **Kernel mode estimation for discrete-space fields** :
  Dabo-Niang et al. (2010), Dabo-Niang et al. (2012) : for functional data and application to bioturbation.

- **Kernel conditional mode and quantile estimation for discrete-space fields** :
  Abdi et al. (2009, 2010), Hallin, Lu et Yu. (2009), Dabo-Niang et Thiam (2010) (real fields), Dabo-Niang, Kaid et Laksaci (2012) (functional data),...

- **Kernel density or regression estimation for continuous-space fields** :
  See Biau (2003), Dabo-Niang and Yao (2007), Bensaid and Dabo-Niang (2007).

Non-parametric regression estimation and prediction for continuously indexed spatial processes

First step : Kernel regression estimation for continuously indexed spatial processes : Dabo-Niang and Yao (2007) (generalization of Biau (2003), Biau and Cadre (2004)).

Second step : Kernel prediction of continuously indexed spatial processes.

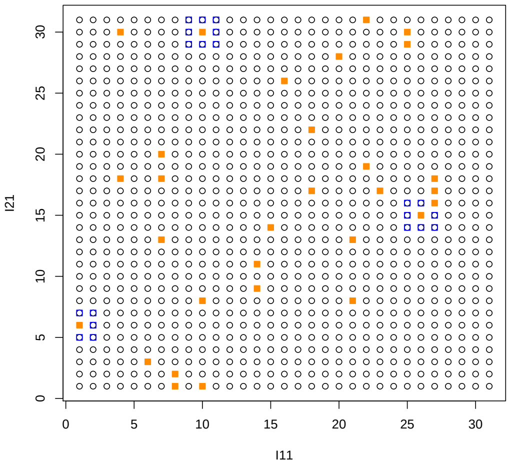
Non-parametric regression estimation and prediction for continuously indexed spatial processes

First step : Kernel regression estimation for continuously indexed spatial processes : Dabo-Niang and Yao (2007) (generalization of Biau (2003), Biau and Cadre (2004)).

Second step : Kernel prediction of continuously indexed spatial processes.

- A natural way is to extend the predictor of Biau and Cadre (2004). But, this last needs a very large sample size.

First non-parametric spatial predictor



How we arrive to consider spatial regression for functional variables

- Let $\mathcal{I}_{\mathbf{T}} = \left\{\mathbf{u} \in \mathbb{R}_+^N \,:\, 0 \leq u_i \leq T_i,\, i = 1, ..., N\right\}$, $(\xi_{\mathbf{t}},\, \mathbf{t} \in \mathbb{R}_+^N)$ be a $\mathbb{R}-$valued strictly stationary random spatial process, assumed to be observed on $\mathcal{O}_{\mathbf{T}} \subset \mathcal{I}_{\mathbf{T}}$, $\mathcal{S}_{\mathbf{T}} = Int(\mathcal{I}_{\mathbf{T}})$ (the interior of $\mathcal{I}_{\mathbf{T}}$), $\mathcal{D}_{\mathcal{S}_{\mathbf{T}}} = \{V \subset \mathcal{S}_{\mathbf{T}},\, V$ is a bounded open set with smooth boundary, $\partial V\}$.
- We want to predict the square integrable value, $\xi_{\mathbf{t}_0}$, at a given non-observed fixed point, $\mathbf{t}_0 \notin \mathcal{I}_{\mathbf{T}}$ admitting a bounded neighborhood,$\mathcal{V}_{\mathbf{t}_0}$ such that $\mathcal{S}_{\mathbf{T}} \cap \mathcal{V}_{\mathbf{t}_0}$ is not empty and belongs to $\mathcal{D}_{\mathcal{S}_{\mathbf{T}}}$.

Spatial prediction : Spatial regression for functional variables

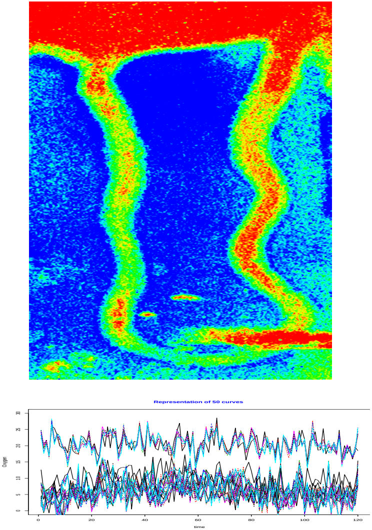


Figure: Oxygen concentration

- Let $\mathcal{V}_0 = \mathcal{V}_{\mathbf{t}_0} - \mathbf{t}_0$ and associate the neighborhood $\mathcal{V}_{\mathbf{t}} = \mathcal{V}_0 + \mathbf{t} = \{\mathbf{u} + \mathbf{t},\, \mathbf{u} \in \mathcal{V}_0\}$ to each $\mathbf{t} \in \mathbb{R}^N$. We set $\mathcal{S}^0 = \{\mathbf{t} \in \mathcal{S}_{\mathbf{T}}, \mathcal{V}_{\mathbf{t}} \in \mathcal{D}_{\mathcal{S}_{\mathbf{T}}}\}$.
- Assume that $(\xi_{\mathbf{t}})$ is an indices-continuous with *germ field Markov property* (see e.g Moura and Goswani (1997) or Pitt and Robeva (2003) for definition).

Prediction of $\xi_{\mathbf{t}_0}$

- Then, the minimum mean-square error prediction of the value $\xi_{\mathbf{t}_0}$ given the boundary data in $\mathcal{I}_{\mathbf{T}}$ is $E(\xi_{\mathbf{t}_0}|\xi_{\mathbf{u}}, \mathbf{u} \in \partial\mathcal{V}_{\mathbf{t}_0})$. This leads us to consider the following associated process :

$$Z_{\mathbf{t}} = (X_{\mathbf{t}}, Y_{\mathbf{t}}) = (\tilde{\xi}_{\mathbf{t}}, \xi_{\mathbf{t}}),\ \mathbf{t} \in \mathbb{R}^N$$

  where $\{\tilde{\xi}_{\mathbf{t}}(\mathbf{u}) = \xi_{\mathbf{u}}, \mathbf{u} \in \partial\mathcal{V}_{\mathbf{t}}\}$ which belongs to the space of continuous and bounded functions, $\mathcal{C}_b(\mathbb{R}^N)$.

The functional predictor

The kernel regression estimator based on $(Z_{\mathbf{t}},\, \mathbf{t} \in \mathcal{I}_{\mathbf{T}})$ is an estimator of a regression with functional predictor :

$$\hat{r}_{\mathbf{T}}(x) = \begin{cases} \dfrac{\int_{\mathcal{I}_{\mathbf{T}}} Y_{\mathbf{t}} K(d(x, X_{\mathbf{t}})_s / h_{\mathbf{T}}) d\mathbf{t}}{\int_{\mathcal{I}_{\mathbf{T}}} K(d(x, X_{\mathbf{t}})_s / h_{\mathbf{T}}) d\mathbf{t}} & \text{if } \int_{\mathcal{I}_{\mathbf{T}}} K(d(x,\, X_{\mathbf{t}})_s / h_{\mathbf{T}}) d\mathbf{t} \neq 0; \\ \frac{1}{\hat{\mathbf{T}}} \int_{\mathcal{I}_{\mathbf{T}}} Y_{\mathbf{t}} d\mathbf{t} & \text{else} \end{cases},$$

$x \in \mathcal{C}_b(\mathbb{R}^N)$, where K is a real valued function defined on $\mathbb{R}_+$ and of integral 1, $d(.,.)_s$ is a semi-norm or a semi-metric on the space $\mathcal{C}_b(\mathbb{R}^N)$, $h_{\mathbf{T}}$ is a bandwidth such that $h_{\mathbf{T}} \geq 0$ and $\lim_{\mathbf{T}\to\infty} h_{\mathbf{T}} = 0$.

- We begin with the regression estimation for discretely indexed functional random fields before considering the continuously indexed problem.

Basic framework

- Consider $Z_{\mathbf{i}} = (X_{\mathbf{i}},\ Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^N}$, $N \geq 1$ a measurable strictly stationary spatial process defined on a probability space $(\Omega,\ \mathcal{A}, \mathbf{P})$ and valued in $\mathcal{E} \times \mathbb{R}$, where $(\mathcal{E},\ d(.,.))$ is a separable infinite dimension semi-metric space with semi-metric $d(.,.)$.
- We assume that the $Z_{\mathbf{i}}$'s have the same distribution as (X, Y), Y is bounded and that the distribution of X satisfies : $\forall t > 0$, $P(X \in B(x,t)) > 0$, where $B(x,t)$ is a ball centered at $x \in \mathcal{E}$ and of radius $t > 0$.
- We observe $X_{\mathbf{i}}$ on $\mathcal{I}_{\mathbf{n}} = \left\{\mathbf{i} \in \mathbb{N}^N\ :\ 1 \leq i_k \leq n_k,\ k = 1, ..., N\right\}$ where, $\mathbf{n} = (n_1, ..., n_N)$.

Spatial prediction : Spatial regression for functional variables

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- We write $\mathbf{n} \to +\infty$ if $\min_{k=1,\ldots,N} n_k \to +\infty$.

General Kernel estimate of r given $Z_{\mathbf{i}},\ \mathbf{i} \in \mathcal{I}_{\mathbf{n}}$

- We define the kernel estimate of r based on $(Z_{\mathbf{i}},\ \mathbf{i} \in \mathcal{I}_{\mathbf{n}})$ as :

General Kernel estimate of r given $Z_{\mathbf{i}},\ \mathbf{i} \in \mathcal{I}_{\mathbf{n}}$

- We define the kernel estimate of r based on $(Z_{\mathbf{i}},\ \mathbf{i} \in \mathcal{I}_{\mathbf{n}})$ as :
- $$r_{\mathbf{n}}(x) = \begin{cases} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} Y_{\mathbf{i}} W_{\mathbf{i},x} & \text{if } \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} W_{\mathbf{i},x} \neq 0; \\ \frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} Y_{\mathbf{i}} & \text{else.} \end{cases}$$

  where

  $$W_{\mathbf{i},x} = \frac{K\left(d(X_{\mathbf{i}}, x) h_{\mathbf{n}}^{-1}\right)}{\sum_{\mathbf{j}\in\mathcal{I}_{\mathbf{n}}} K\left(d(X_{\mathbf{j}}, x) h_{\mathbf{n}}^{-1}\right)}.$$

General Assumptions

We suppose that (such as in Tran (1990), Dabo-Niang and Yao (2007) and Ferraty and Vieu (2006)) that : (in what follows C will denote a positive constant)

- We assume also that for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^N$ the joint probability distribution $\nu_{\mathbf{i},\mathbf{j}}$ of $X_{\mathbf{i}}$ and $X_{\mathbf{j}}$ satisfies

$$\exists \epsilon_1 \in (0,1],\ \nu_{\mathbf{i},\mathbf{j}}\left(B(x,h_{\mathbf{n}}) \times B(x,h_{\mathbf{n}})\right) = (p^x_{h_{\mathbf{n}}})^{1+\epsilon_1}, \qquad (1)$$

where $p^x_{h_{\mathbf{n}}} = P(X \in B(x,h_{\mathbf{n}}))$.

- HK_1 : there exist two constants $0 < C_1 < C_2 < \infty$:

$$C_1 I_{[0,1]} \leq K \leq C_2 I_{[0,1]}.$$

where $I_{[0,1]}$ is the indicator function in $[0,1]$.
or

- The support of K is $[0,1]$, the derivative K' of K exists and satisfies

$$-\infty < C_1 \leq K' \leq C_2 < 0 \text{ and } -\exists C > 0, \exists \varepsilon_0 > 0, \forall \varepsilon < \varepsilon_0,$$

$$\int_0^{\varepsilon} \mu(B(x,z))\, dz > C\varepsilon\mu(B(x,\varepsilon)).$$

- HF_1 - The regression function r is continuous at $x \in \mathcal{E}$.

Spatial prediction : Spatial regression for functional variables

For uniform rate of convergence

We consider a set $\mathcal{C}$ such that $\mathcal{C} \subset \mathcal{C}_{\mathbf{n}}$ where $\mathcal{C}_{\mathbf{n}} = \bigcup_{k=1}^{d_{\mathbf{n}}} B(t_k, \rho_{\mathbf{n}})$ (note that such set $\mathcal{C}_{\mathbf{n}}$ can always be built), $d_{\mathbf{n}} > 0$ is some integer, $t_k \in \mathcal{E}$, $k = 1, ..., d_{\mathbf{n}}$, and $\rho_{\mathbf{n}} > 0$. We assume that $\mathcal{C}$ is such that :

- H_1 - $\sup_{x \in \mathcal{C}} p^x_{h_{\mathbf{n}}} = \Gamma(h_{\mathbf{n}}) > 0$ exists.

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- H_2 - $d_{\mathbf{n}} = \widehat{\mathbf{n}}^{\beta}$ and
  $\rho_{\mathbf{n}} \leq (h_{\mathbf{n}})^{\kappa} \left(\log \widehat{\mathbf{n}} / (\widehat{\mathbf{n}} \Gamma(h_{\mathbf{n}}))\right)^{1/2}$, $\beta > 0 \quad \kappa > 1$.

For uniform rate of convergence

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- HK_2 : K is a Lipschitz function.

For uniform rate of convergence

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- HF_2 - We assume in the following theorem that r is a Lipschitz function.

$\alpha-$Mixing condition

We assume that :

- $\exists\ \varphi(t) \downarrow 0$ as $t \to \infty$, such that whenever $E,\ E^{'}$ subsets of $\mathbb{N}^N$ with finite cardinals,

$$\alpha\left(\mathcal{B}(E), \mathcal{B}\left(E^{'}\right)\right) = \sup_{B\in\mathcal{B}(E),\, C\in\mathcal{B}\left(E^{'}\right)} \left|\mathbf{P}(B\cap C) - \mathbf{P}(B)\,\mathbf{P}(C)\right|$$

$$\leq \psi\left(\text{Card}(E), \text{Card}\left(E^{'}\right)\right)\varphi\left(\text{dist}\left(E, E^{'}\right)\right), \qquad (2)$$

where $\mathcal{B}(E)$(*resp.* $\mathcal{B}\left(E^{'}\right)$) is the Borel σ-field generated by $(X_{\mathbf{i}},\ \mathbf{i} \in E)$ (*resp.* $\left(X_{\mathbf{i}},\ \mathbf{i} \in E^{'}\right)$), Card$(E)$ (*resp.* Card$\left(E^{'}\right)$) the cardinality of E (*resp.* $E^{'}$), dist$\left(E, E^{'}\right)$ the Euclidean distance between E and $E^{'}$ and $\psi : \mathbb{N}^2 \rightarrow \mathbb{R}^{+}$ is a symmetric positive function nondecreasing in each variable.

Spatial prediction : Spatial regression for functional variables

It will be assumed that :

- ψ satisfies either

$$\psi\,(n,m) \leq C \min\,(n,m)\,, \qquad \forall n, m \in \mathbb{N} \tag{3}$$

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$$\psi(n,m) \leq C \min(n,m), \qquad \forall n,m \in \mathbb{N} \tag{3}$$

- or

$$\psi(n,m) \leq C(n+m+1)^{\widetilde{\beta}}, \qquad \forall n,m \in \mathbb{N} \tag{4}$$

for some $\widetilde{\beta} \geq 1$ and some $C > 0$.

Spatial prediction : Spatial regression for functional variables

Polynomial rate

We will consider the case where $\varphi(i)$ tends to zero at a polynomial rate, that is :

$$\varphi(i) \leq Ci^{-\theta}, \text{ for some } \theta > 0. \tag{5}$$

The results for exponential mixing case can be deduce easily from the one of polynomial case.

Consistency results

Denote

$$\theta_1^* = \frac{2N-\theta}{4N-\theta},\ \theta_2^* = \frac{-\theta+N}{N(2\widetilde{\beta}+3)-\theta},$$

$$\theta_3^* = \frac{-\theta+2N}{-\theta+2N(\beta+2)},\ \theta_4^* = \frac{-\theta+N}{N(3+2\beta+2\widetilde{\beta})-\theta}.$$

Denote $g(\mathbf{n}) = \prod_{i=1}^{N}(logn_i)(loglogn_i)^{1+\epsilon}$, $\epsilon > 0$.

Strong Consistency results

Theorem

Under the assumptions HF_1, HK_1, *(1), (2), (5),* $\widehat{\mathbf{n}} p^x_{h_\mathbf{n}} / \log \widehat{\mathbf{n}} \to \infty$ *and if the mixing verifies :*

- *the conditions (3),* $\theta > 4N$ *and*
$$\left(\widehat{\mathbf{n}} \left(\frac{p^x_{h_\mathbf{n}}}{\log \widehat{\mathbf{n}}} \right)^{\theta_1^*} g(\mathbf{n})^{\frac{2N}{4N-\theta}} \right)^{\frac{4N-\theta}{2N}} \to \infty$$
or
- *the conditions (4),* $\theta > N(3+2\widetilde{\beta})$ *and*
$$\left(\widehat{\mathbf{n}} \left(\frac{p^x_{h_\mathbf{n}}}{\log \widehat{\mathbf{n}}} \right)^{\theta_2^*} g(\mathbf{n})^{\frac{2N}{N(2\widetilde{\beta}+3)-\theta}} \right)^{\frac{N(2\widetilde{\beta}+3)-\theta}{2N}} \to \infty$$

then,

$$|r_\mathbf{n}(x) - r(x)| \text{ converges almost surely to } 0. \qquad (6)$$

Theorem

Under conditions HK_2 *and* HF_2 *and if* $\widehat{\mathbf{n}}\Gamma(h_{\mathbf{n}})/\log\widehat{\mathbf{n}} \to \infty$, H_1, H_2 *and*

- *the conditions (3),* $\theta > 2N(\beta+2)$ *and* $\left(\widehat{\mathbf{n}}(\Gamma(h_{\mathbf{n}})/\log\widehat{\mathbf{n}})^{\theta_3^*}(g(\mathbf{n}))^{\frac{2N}{-\theta+2N(\beta+2)}}\right) \to \infty$ *or*
- *the conditions (4),* $\theta > N(3+2\beta+2\widetilde{\beta})$ *and* $\left(\widehat{\mathbf{n}}(\Gamma(h_{\mathbf{n}})/\log\widehat{\mathbf{n}})^{\theta_4^*}(g(\mathbf{n}))^{\frac{2N}{-\theta+N(2\beta+2\widetilde{\beta}+3)}}\right) \to \infty$

hold, we have

$$\sup_{x\in\mathcal{C}} |r_{\mathbf{n}}(x) - r(x)| = \mathcal{O}\left(h_{\mathbf{n}} + \sqrt{\frac{\log\widehat{\mathbf{n}}}{\Gamma(h_{\mathbf{n}})\widehat{\mathbf{n}}}}\right), \quad a.s. \qquad (7)$$

We simulate the model model $(X_{i,j}, Y_{i,j})$, $1 \leq i,\, j \leq 30$ with :

$$X_{i,j}(t) = A_{i,j} * (t - 0.5)^2 + B_{i,j}$$

and

$$Y_{i,j} = 4A_{i,j}^2 + \varepsilon_{i,j}.$$

where $(A_{i,\,j})$ and $(B_{i,\,j})$ are Gaussian random fields and $\varepsilon_{i,j}$ is normally distributed with zero mean.

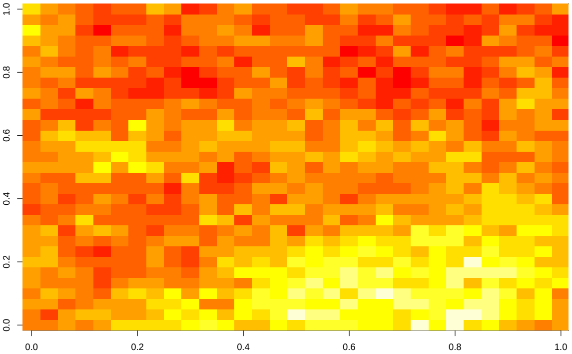


Figure: The random field A

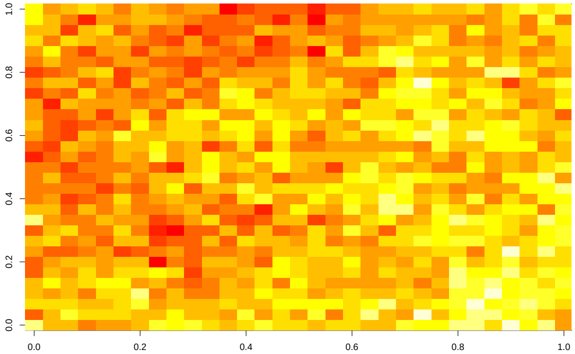


Figure: The ramdom field B

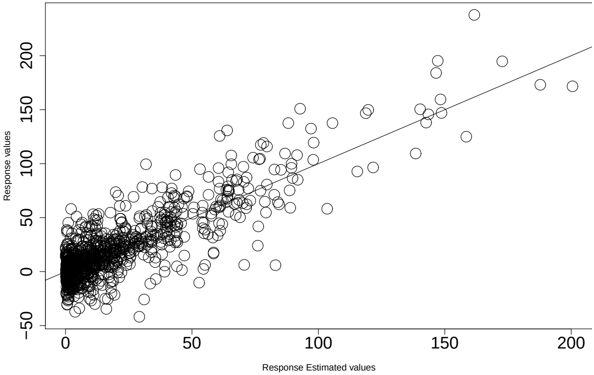


Figure: DRY with knn=20 and hopt

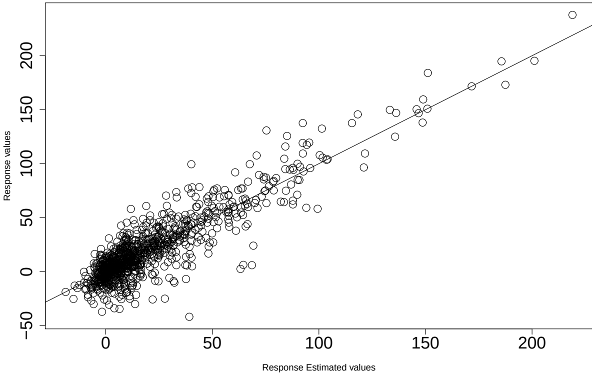
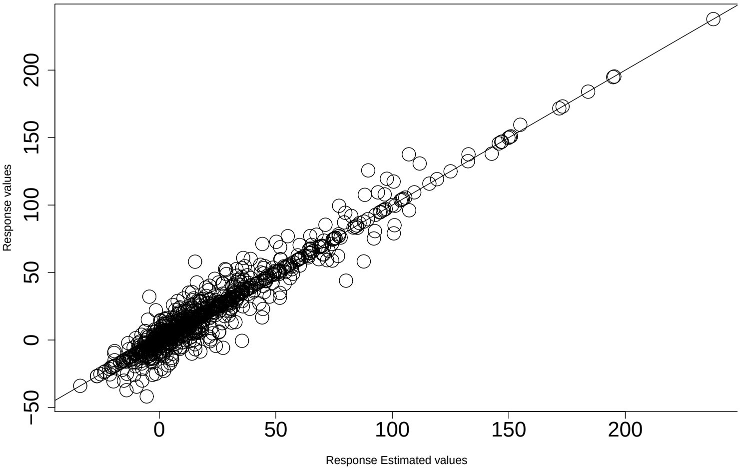


Figure: FV with hopt

Spatial prediction : Spatial regression for functional variables



Spatial prediction : Spatial regression for functional variables

	MSE	RMSE
DRY with hopt	67.51	0.25×10^{-3}
FV with hopt	304.87	0.51
DRY with hopt of fv	208.61	0.43

Table: The MSE and RMSE

Estimation of conditional quantiles

- $F(\cdot|x)$, distribution function $Y_{\mathbf{i}}$ given $X\mathbf{i} = x$, $f(\cdot|x)$ the density

Estimation of conditional quantiles

- $F(\cdot|x)$, distribution function $Y_{\mathbf{i}}$ given $X\mathbf{i} = x$, $f(\cdot|x)$ the density
- For $p \in]0, 1[$ and $x \in \mathbb{R}^d$ fixed, $\mu_p(x)$ the quantile of order p of $F(\cdot|x)$: $F(y|x) = p$.

Estimation of conditional quantiles

- $F(\cdot|x)$, distribution function $Y_{\mathbf{i}}$ given $X\mathbf{i} = x$, $f(\cdot|x)$ the density
- Or
  $\mu_p(x) = \arg\min_{\theta \in \mathbb{R}} E\left[(2p-1)(Y-\theta) + |Y-\theta| \;|X = x\right]$.

Estimation of $\mu_p(x)$

$$F^1_{\mathbf{n}}(y|x) = \frac{\psi^1_{\mathbf{n}}(x,y)}{f_{\mathbf{n}}(x)} \mathbb{I}_{\{f_{\mathbf{n}}(x)\neq 0\}}.$$

avec $\psi^1_{\mathbf{n}}(x,y) = \frac{1}{\widehat{\mathbf{n}} h^{d+1}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} K\left(\frac{x - X_{\mathbf{i}}}{h}\right) \int_{-\infty}^{y} w\left(\frac{z - Y_{\mathbf{i}}}{h}\right) dz$

$f_{\mathbf{n}}(x) = \frac{1}{\widehat{\mathbf{n}} h^{d}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} K\left(\frac{x - X_{\mathbf{i}}}{h}\right)$, K and w are kernels, $h = h(\mathbf{n})$, $h \to 0$ when $\mathbf{n} \to \infty$.

For x fixed, an estimator of $\mu_p(x)$ denoted $\mu^1_{p,\mathbf{n}}(x)$ is such that $F^1_{\mathbf{n}}(z|x) = p$.

The local linear alternative :

$$\mu_{p,\mathbf{n}}^2(x) = \arg\min_{\theta\in\mathbb{R}} \sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} \left(|Y_\mathbf{i}-\theta| + (2p-1)(Y_\mathbf{i}-\theta)\right) K\left(\frac{x-X_\mathbf{i}}{h}\right). \quad (8)$$

We have (8)

$$\mu_{p,\mathbf{n}}^2(x) = \inf\left\{t\in\mathbb{R},\quad F_\mathbf{n}^2(t|x)\geq p\right\}, \quad (9)$$

where $F_\mathbf{n}^2(\cdot|x)$ is the estimator $F(\cdot|x)$:

$$F_\mathbf{n}^2(y|x) = \frac{\psi_\mathbf{n}^2(x,y)}{g_\mathbf{n}(x)} \mathbb{I}_{\{g_\mathbf{n}(x)\neq 0\}}.$$

$$\text{avec} \quad \psi_\mathbf{n}^2(x,y) = \frac{1}{\widehat{\mathbf{n}}h^d} \sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} K\left(\frac{x-X_\mathbf{i}}{h}\right) \mathbb{I}_{\{Y_\mathbf{i}\leq y\}}$$

Conclusion

- Non-parametric modeling (by regression, conditional quantile, mode,... are alternative tools to the kriging).
- Strong Markov hypothesis : Wang et al. (2012),...
- Strict stationarity
- Conditions on h_n
- Spatio-temporal

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